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Analysis of Quantized Double Auctions with Application to Competitive Electricity Markets

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Abstract—In recently proposed electricity markets, price-based competitive behaviours of power suppliers (i.e., generators), energy service providers and large users (i.e., consumers) have been formulated using various auction algorithms (see Post *et al*., 1995; Wolfram, 1998; Dekrajangpetch and Shebl, 2000; Nicolaisen *et al*., 2001; Swider and Weber, 2007). In this paper, quantized Progressive Second Price (PSP) auction algorithms are presented for competitive electricity systems, especially for short-run electric power markets. In Jia and Caines (2008, 2010), two quantized PSP auction algorithms were introduced and analyzed for demand markets, which are called, respectively, the Aggressive-Defensive Quantized PSP (ADQ-PSP) algorithm and the Unique-limit Quantized PSP (UQ-PSP) algorithm. Here we first present an algorithm combined with ADQ-PSP and UQ-PSP features, and apply it to a double power auction system where competition on both power generators and energy service providers (and/or large users) is considered. Double auctions are formulated in this work as two single-sided quantized auctions which depend upon joint market quantities and price constraints. The extended algorithm inherits the performance properties of ADQ-PSP and UQ-PSP in terms of both the social welfare maximization and the rapid convergence rate.

Keywords Decentralized decision, dynamical double auctions, multi-agent systems, competitive markets.

1. INTRODUCTION

Pioneering electricity industry deregulations have been performed or considered in the United Kingdom, New Zealand, Germany, America, to name but a few. The key motivation of such restructuring is to design efficiently decentralized rules in power systems to substitute for the traditional centralized regulation such that the true marginal costs of power generators (and/or large consumers) can be revealed by decentralized decisions, and in addition, social welfare will be improved by the competition among the agents. Consequentially, auction based methods (see Post *et al*., 1995; Wolfram, 1998; Dekrajangpetch and Shebl, 2000; Nicolaisen *et al*., 2001; Swider and Weber, 2007) have been applied as competitive and strategic pricing on such electricity system models.

Power auctions can be classified by different criteria: (i) based upon time periods and bid dimensions, centralized daily commitment (power pool) auctions (see Madrigal and Quintana, 2001; Motto *et al*., 2002) and single period commodity auctions (Motto and Galiana, 2002) are applied in different countries; (ii) at the different stages of markets, wholesale auctions and retail auctions are separately considered; (iii) single-sided (Wolfram, 1998) and double-sided auctions (Nicolaisen *et al*., 2001) are exercised due to agents' market power; (iv) uniform and discriminatory pricings (Nicolaisen *et al*., 2001) are also adopted in these auction algorithms.

Here we focus on a single-period competitive electricity market model and mainly consider the competition among power suppliers (i.e., sellers) and energy service providers (and/or large users) (i.e., buyers). Quantized strategies for this model are established based upon our previous work on demand auctions (Jia *et al*., 2009; Jia and Caines, 2010) and the fact that quantization is meaningful in practice. As a result, the specified double-sided, uniform-pricing, open-bid, and iteratively dynamical auction is shown to have desirable properties: (i) acceptable pricing can be achieved simply and rapidly and (ii) the market participants (i.e., power generators, service providers and large users) have an incentive to ensure market efficiency.

In (Jia and Caines, 2010), so-called Aggressive-Defensive Quantized Progressive Second Price (ADQ-PSP) auction

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algorithm and Unique-limit Quantized Progressive Second Price (UQ-PSP) auction algorithm were first presented for dynamic market-pricing in communication networks and social networks. These algorithms dynamically allocate a divisible resource to agents capable of exerting market power to generate successive price and quantity bids on the demand side. Subject to a quantization assumption on the bid price set, and considering agent populations with randomly distributed demand functions, it was proved in (Jia and Caines, 2010) that the nonlinear dynamics induced by the ADQ-PSP algorithm converges extremely rapidly with high probability to quantized (Nash) equilibria with a common price for all agents. On the other hand, the UQ-PSP algorithm was considered as a modification of ADQ-PSP in order to improve social efficiency but at the cost of a less rapid convergence rate; for this VCG-like algorithm, (i) the associated dynamical auction systems converge to a unique (ϵ-Nash) equilibrium (i.e., independent of initial data); (ii) the limit resource allocation is efficient (in the sense of the optimization of the summed individual valuation functions) up to a quantized level under mild assumptions of demand functions.

In this paper we consider a double auction case, where agents (i.e., both sellers and buyers) in a power market are assumed to have market power, and their strategies not only influence their peers' behaviours but also influence the dynamics of the other side of the market (see Wilson, 1985; Gjerstad and Dickhaut, 1998, Luckock, 2003). Each agent here is assumed to apply a quantized scheme derived from the features of the ADQ-PSP and UQ-PSP algorithms. Subject to this quantization assumption (a meaningful constraint, see Sotheby's, 2010), the dynamical double auction system, consisting of recursive quantized bids and market information from both demand and supply sides, is shown to converge to an order-two orbit. The orbit consists of the two quantized prices defining the smallest price set approximating the competitive equilibrium price under the quantized framework. Social efficiency is hence achieved modulo to a quantized level.

This paper is organized as follows.

- In Section 2, we formulate the pricing of short-run competitive electric power market model as a progressive second price double auction. The public and private information in the market is clarified, and agents' utility functions are indicated according to the allocation rule and cost function.
- Quantized strategies and the associated dynamical double auction system are established in Section 3.
- The convergence and efficiency of such a system are then analyzed in Section 4.
- Numerical simulations are given in Section 5. It is shown that the properties of rapid convergence and approximate efficiency of the algorithm help to restructure the daily or hourly electricity competition.
- Finally, we summarize our results in Section 6 together with some remarks concerning future research.

2. PROBLEM FORMULATION

In our previous work (Jia and Caines, 2010), it was assumed that sellers in the competitive demand market are price-taking and they do not influence the limit price of the dynamical demand auction. We will relax that assumption here and study an alternative situation where competitive behaviours on both the supply-side and the demand-side of the power market are considered. This problem is formulated as a double auction where both buyers and sellers make two-dimensional bids simultaneously. Buyers and sellers recursively update their strategies to maximize their own utility by observing the current market information. In contrast to the single-sided auction, each buyer's (respectively, seller's) bid not only influences its opponents' behaviours, but also influences the dynamics on the supply (respectively, demand) side, and hence has an impact on the market total power quantity C and the lower-bound (respectively, upper-bound) of bid prices. Consistent with our overall theoretical framework and for practical meaningfulness, we retain the basic quantization assumption on bid prices.

In this section, we first introduce the buyer and seller bidding profiles into one static model, and then present the notions of matched prices and quantities and potential quantity. Competition in the double auction will be formulated as two single-sided auctions which depend upon these joint market quantities and price constraints. That is to say, we describe such competition as a non-cooperative game and introduce the coupling parameters between both sides of the market in a static model. Next, in Section 3, we construct a dynamical quantized double auction system which consists of two coupled recursive subsystems.

Specifically, in a non-cooperative power market, N power suppliers (i.e., sellers) produce divisible electricity resource to satisfy the requirement of M large users (i.e., buyers). Each seller makes a two-dimensional bid $s_j = (ps_j, qs_j)$, $1 \leq j \leq N$, to a market operator (auctioneer), where qs_j is the quantity the seller desires to provide and ps_j is the unit-price at which the seller wishes to sell qs_j. Here we assume $ps_j \leq p_c < \infty$, for all $1 \leq j \leq N$, where $p_c > 0$ is defined to be a *threshold price*, i.e., the highest price the market can bear. Correspondingly, each buyer also makes a two-dimensional bid $b_i = (pb_i, qb_i)$, $1 \leq i \leq M$, to the market operator, where qb_i is the quantity the buyer desires and pb_i is the unit-price the buyer wishes to pay for qb_i.

We define:

- *Buyer bidding profile:* $b \triangleq [b_i]_{1\leq i\leq M}$; and the bidding profile of any buyer B_i's opponents: $b_{-i} \triangleq b\backslash\{b_i\}$.
- *Seller bidding profile*: $s = [s_j]_{1\leq j\leq N}$; and the bidding profile of any seller S_j's opponents: $s_{-j} = s\backslash\{s_j\}$.

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- The *buyer bid price function (BPF)* is a right-continuous (i.e., continuous from the right) function defined to be:

$$\mathcal{B}(q,b) = \inf\left\{ p \geq 0 : \sum_{pb_i > p} qb_i \leq q \right\}, \tag{1}$$

i.e., a mapping from quantities q and bidding profiles b to prices p.

The *seller bid price function (SPF)* is a right-continuous function defined to be:

$$\mathcal{S}(q,s) = \sup\left\{ p_c \geq p \geq 0 : \sum_{ps_j < p} qs_j \leq q \right\}. \tag{2}$$

These two bid price functions are introduced in order to describe the public bidding information on both sides of a competitive market, which consist of the buyer bidding profile (ordered w.r.t. bid prices from high to low since the higher bid price for a buyer implies the higher priority to obtain the resource) and the seller bidding profile (ordered w.r.t. bid prices from low to high since the lower bid price for a seller implies the higher priority to sell the resource), respectively.

- The *matched prices* pb^* and ps^* are, respectively, the buyer bid price and the seller bid price such that

$$pb^*(b,s) = \mathcal{B}(q^*, b), \tag{3}$$

$$ps^*(b,s) = \mathcal{S}(q^*, s). \tag{4}$$

where $q^* = \sup_q\{\mathcal{B}(q,b) \geq \mathcal{S}(q,s)\}$, i.e., the superior quantity such that the BPF function is not less than the SPF function. Buyers bidding a price higher than or equal to ps^* and sellers bidding a price lower than or equal pb^* are called *matched agents* (see Figure 1). One may check that $pb^* \geq ps^*$. Here the "matched" concept is applied in order to describe the relation between two competitive sides of the market. This notion delineates subsets of the sellers and the buyers (determined by the matched prices respectively)

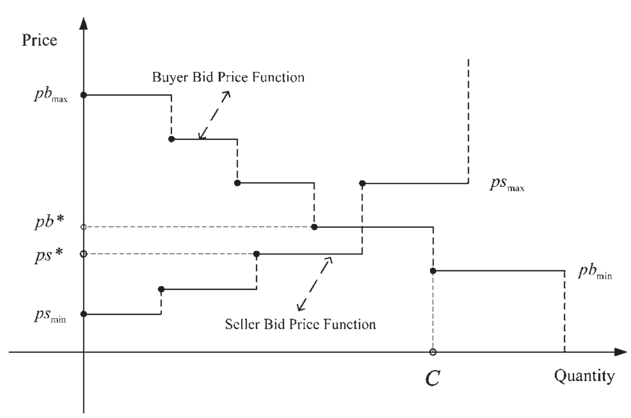

Figure 1. Public bid information in a quantized double auction

which can trade positive quantity of goods with each other. In this paper we assume that agents in the market are rational and therefore, they shall make strategic bids such that they are matched. The total potentially traded quantity is then analyzed in terms of the notion of matched quantities which are defined as follows.
Set the *matched quantities*

$$Cb^*(b,s) = \sum_{pb_i \geq ps^*(b,s)} qb_i, \tag{5}$$

$$Cs^*(b,s) = \sum_{ps_j \leq pb^*(b,s)} qs_j, \tag{6}$$

then the *potential quantity* C is such that

$$C = C(b,s) = \begin{cases} \min\{Cb^*(b,s), Cs^*(b,s)\}, & \text{if } pb^* = ps^*, \text{and } |B_p| = 1; \\ \max\{Cb^*(b,s), Cs^*(b,s)\}, & \text{otherwise,} \end{cases} \tag{7}$$

where B_p is the bid price set, i.e., $B_p \triangleq \{ps_j, 1 \leq j \leq N\} \cup \{pb_i, 1 \leq i \leq M\}$. Here C is specified in two different situations: (i) When the bid prices of the sellers do not agree with those of the buyers, i.e., $pb^* > ps^*$ or $pb^* = ps^*$ and $|B_p| > 1$, the total available quantity considered by both sides at the next iteration is chosen as the larger matched quantity. This assumption reflects the market potentiality for higher matched quantity level and it encourages more resource to be produced or required in the market before the agreement is achieved. (ii) When there exists a unique matched price, i.e., $pb^* = ps^*$ and $|B_p| = 1$, C is assumed to be the smaller accumulated bid quantity on both sides, which corresponds to the actual matched quantity if the auction stops at this iteration.

Figure 1 describes the relationship between bid price functions (BPF and SPF) and matched information (prices pb^*, ps^* and quantity C).

By introducing the matched quantities and prices, we can decompose the considered double-sided competition into two dependent *dual* subsystems: the *progressive second price (PSP) supply* auction and the *progressive second price demand* auction.

I. Competition among sellers (PSP supply auction)

- Given a potential quantity C, the *seller market-price function* (SMF) of Seller $S_j, 1 \leq j \leq N$, is a left-continuous function defined to be:

$$Ps_j(q, s_{-j}) = \sup_p\left\{ p_c \geq p \geq 0 : C - \sum_{ps_i < p, i \neq j} qs_j \geq q \right\}, \tag{8}$$

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which is interpreted as the maximum bid price the seller asks in order to supply the quantity q given the opponents' profile s_{-j}, the quantity C and the threshold price p_c. The function is only meaningful for $q > 0$. Similarly we define its inverse function Qs_j as follows:

$$Qs_j(p, s_{-j}) = \left[C - \sum_{ps_i < p, i \neq j} qs_j\right]^+, \tag{9}$$

which means the minimum supply quantity at a bid price of p given s_{-j}.

- The private *supply function* $\sigma_j : \mathcal{R}^+ \to \mathcal{R}^+$ of Seller $S_j, 1 \le j \le N$, is non-decreasing and continuous. Here a supply function represents the amount of resource that a supplier prefers to sell at various prices (i.e., the marginal cost function in Green and Newbery, 1992). Denote $Y_j(q) = \int_0^q \sigma_j(t)\mathrm{d}t, 1 \le j \le N$, as the *production cost function* and $I_j = \sigma_j^{-1}, 1 \le j \le N$, as the inverse supply function. The *potential valuation function* is defined to be $V_j(q) = p_c \cdot q - Y_j(q),\ 1 \le j \le N$. It is noted that these functions are privately associated with S_j, and are unknown to all the other agents and the auctioneer. For simplicity of analysis, in the following discussion, we assume that:

Hypothesis 2.1 *(Elasticity Supply).*
Each σ_j satisfies an elasticity assumption: there exists $\gamma_j > 0$, such that for all $0 \le q' < q \le C$, $\sigma_j(q') > 0$ implies $\sigma_j(q) - \sigma_j(q') \ge \gamma_j(q - q')$.

- The allocation rule for sellers is defined as follows:

$$a_j(s) = \min\left\{qs_j, \frac{qs_j \cdot Qs_j(ps_j, s_{-j})}{\sum_{i:ps_i = ps_j} qs_i}\right\} \tag{10}$$

$$c_j(s) = \sum_{i \neq j}(p_c - ps_i)[a_i(0; s_{-j}) - a_i(s_j; s_{-j})], \tag{11}$$

where a_j denotes the quantity Seller S_j sells at a bid price ps_j given the opponents' bidding profile s_{-j}, i.e., the minimum of S_j's bid quantity qs_j and the available market quantity at the bid price ps_j. The cost to S_j shall include two parts: the production cost $Y_j(a_j(s))$ and the opportunity cost c_j. c_j is introduced based upon the concept of the *exclusion compensation principle* (Lazar and Semret, 1999) and the Vickrey-Clarke-Groves (VCG) mechanism (Makowski and Ostroy, 1987); it represents the potential difference in revenue between that contributed by all the other sellers distinct from S_j when (i) S_j is absent from the auction and (ii) S_j participates in the auction. The Progressive Second Price auction is named after this feature (Lazar and Semret, 1999).
- Seller S_j's utility is defined to be

$$\begin{aligned} u_j(s) &= p_c \cdot a_j(s) - Y_j(a_j(s)) - c_j(s) \\ &= V_j(a_j(s)) - c_j(s), \end{aligned} \tag{12}$$

i.e., the potential valuation minus the opportunity cost at the allocated quantity $a_j(s)$.

For the sake of simplicity, here we assume that the sellers (and buyers) in the market do not have budget constraints. Then given the opponents' bidding profile s_{-j} and the potential quantity C, the best reply (i.e., maximizing $u_j(s)$) of a seller under the framework above can be chosen as follows:

Lemma 2.2. *Subject to Hypothesis 2.1, given s_{-j} and C, Seller S_j's best response is obtained by $s_j = (ws_j, vs_j)$, where*

$$\begin{aligned} vs_j &= \inf\{q \ge 0 : \sigma_j(q) > Ps_j(q, s_{-j})\}, \\ ws_j &= \sigma_j(vs_j). \end{aligned} \tag{13}$$

The proof of Lemma 2.2 is similar to that presented in (Lazar and Semret, 1999) in which the dual problem on a demand-side auction is studied. Due to space constraints, we omit it here. It is noted that the best strategy (ws_j, vs_j) is *truth-telling*, i.e., the bid is chosen truthfully according to σ_j.

The relationship between Seller S_j's supply function, its market price function and its best strategy given in (13) is shown in Figure 2, where purely for simplicity of portrayal, it is assumed that σ_j is linear.

II. Competition among buyers (PSP demand auction, Lazar and Semret, 1999)

- The *buyer market-price function (BMF)* of Buyer B_i, $1 \le i \le M$, is defined as:

$$Pb_i(q, b_{-i}) = \inf\left\{p \ge 0 : C - \sum_{pb_j > p, j \neq i} qb_j \ge q\right\},$$

where C is determined by (7) and it was a fixed value for demand auctions in (Jia and Caines, 2010; Lazar and

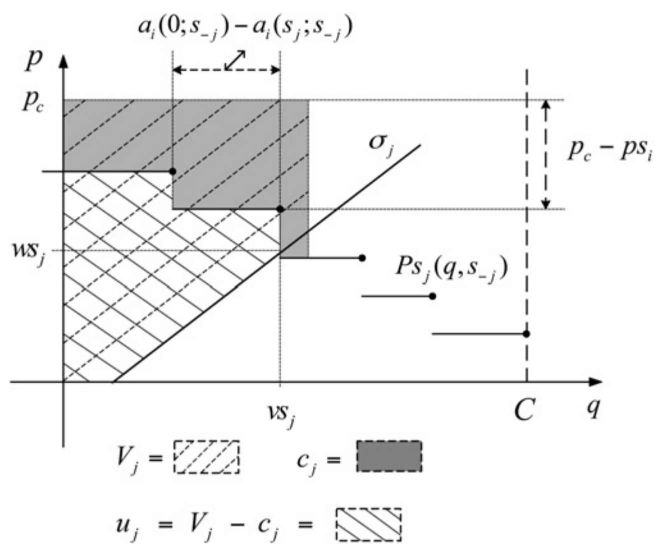

Figure 2. Market information and the best strategy

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Semret, 1999). Its inverse function is:

$$Qb_i(p, b_{-i}) = \left[C - \sum_{pb_j > p, j \neq i} qb_j \right]^+ .$$

- The private *demand functions* $\delta_i(\cdot) : \mathcal{R}^+ \to \mathcal{R}^+$, $1 \le i \le M$, of buyers are assumed to be decreasing and continuous. Denote $D_i = \delta_i^{-1}$ as the inverse demand function and $Z_i(q) = \int_0^q \delta_i(z)\,\mathrm{d}z$ as the valuation (or reward) functions of buyers. The valuation function Z_i and the demand function δ_i reflect the intrinsic value and marginal value of a certain amount of resource for a buyer B_i, respectively. It is noted here that (i) these functions are privately associated with B_i, and are unknown to all the other agents and the auctioneer and (ii) for convenience of reference, we assume that the demand function δ_i is a mapping from quantity q to price p and so it corresponds to the "inverse demand function" commonly used in economics literature (see, e.g., Samuelson and Marks, 2003; Krugman and Wells, 2005).

Hypothesis 2.3. *(Elasticity demand, Lazar and Semret, 1999)*

We assume that $Z_i(0) = 0$, $1 \le i \le M$, *and for any* $C \ge q > q' \ge 0$, *there exists* $\gamma_i > 0$, *such that* $\delta_i(q) > 0$ *implies* $\delta_i(q) - \delta_i(q') \le -\gamma_i(q - q')$.

- The *allocation rule* (first presented in Tuffin, 2002) for buyers is then defined as follows:

$$a_i(b) = \min\left\{ qb_i, \frac{qb_i \cdot Qb_i(pb_i, b_{-i})}{\sum_{j:pb_j = pb_i} qb_j} \right\}, \tag{14}$$

$$c_i(b) = \sum_{j \neq i} pb_j [a_j(0; b_{-i}) - a_j(b_i; b_{-i})], \tag{15}$$

where $a_i(b)$ denotes the quantity Buyer B_i obtains by a bid price pb_i when the opponent buyers bid b_{-i} and the potential quantity is C, and the charge to Buyer B_i is denoted as $c_i(b)$.
- Buyer B_i's *utility* (Lazar and Semret, 1999) is defined to be

$$u_i(b) = Z_i(a_i(b)) - c_i(b). \tag{16}$$

In a similar way, we can find buyers' static best strategies $b_i = (wb_i, vb_i)$, given b_{-i} and C,

$$\begin{aligned} vb_i &= \sup\{q \ge 0 : \delta_i(q) > Pb_i(q, b_{-i})\}, \\ wb_i &= \delta_i(vb_i), \end{aligned} \tag{17}$$

which was first described in (Lazar and Semret, 1999).

3. QUANTIZED STRATEGIES AND DYNAMICAL AUCTION SYSTEM

By using the definitions above, one may check that the strategies (vs_j, ws_j), $1 \le j \le N$, on the supply-side specified in Lemma 2.2 and the strategies (vb_i, wb_i), $1 \le i \le M$, on the demand-side given in (17) are still the best replies with respect to the given information $\{b, s\}$ (and $C(b, s)$) (i.e., they are dominant strategies). In this section, we consider agents' behaviours in such a double(-sided) auction but subject to a quantized pricing constraint (that is to say, all agents' bid prices should be chosen from a finite price set B_p^0). This is motivated by that fact that quantized pricing has many practical examples (see, e.g., Sotheby's, 2010). For a quantized double auction, each agent (a buyer or seller) at each biding iteration makes a (quantized) strategic bid to maximize its own utility. Such rational strategies may not be unique. In the following we develop a family of quantized strategies and show that they are γ-best replies subject to quantization. These strategies are extended from our previous work on the ADQ-PSP algorithm and the UQ-PSP algorithm (Jia and Caines, 2010).

Quantized strategies for PSP double auctions: Given the bidding profiles $\{b, s\}$, the supply function σ_i, the potential quantity $C = C(b, s)$ and a quantized price set B_p^0, the quantized strategy $s_j = (ps_j, qs_j)$ for Seller S_j, $1 \le j \le N$, is as follows:

$$\begin{aligned} ps_j &= T_s(b, s) = \min\left\{ \min_{1 \le i \le M} \{ pb_i \ge ps^* \}, T_1(b, s) \right\}, \\ qs_j &= I_j(ps_j) = \sigma_j^{-1}(ps_j), \end{aligned} \tag{18}$$

where

$$T_1(b, s) \triangleq \begin{cases} \min\{p \in B_p^0; p \ge ws_j\}, & \text{if } \sum_{1 \le i \le N} qs_i \le C, \\ \max\{p \in B_p^0; p < Ps_j(vs_j, s_{-j})\}, & \\ \qquad \text{if } ws_j \neq p_{\min} \text{ and } \sum_{1 \le i \le N} qs_i > C, & \\ p_{\min}, \quad \text{if } ws_j = p_{\min} \text{ and } \sum_{1 \le i \le N} qs_i > C, & \end{cases} \tag{19}$$

$p_{\min} \triangleq \min B_p^0$, and the best strategy (vs_j, ws_j) is given in Lemma 2.2.

Similarly, given $\{b, s, B_p^0, \delta_i\}$, the quantized strategy $b_i = (pb_i, qb_i)$ for Buyer B_i, $1 \le i \le M$, is defined as

$$\begin{aligned} pb_i &= T_b(b, s) = \max\left\{ \max_{1 \le j \le N} \{ ps_j \le pb^* \}, T_2(b, s) \right\}, \\ qb_i &= D_i(pb_i) = \delta_i^{-1}(pb_i), \end{aligned} \tag{20}$$

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where

$$T_2(b,s) \triangleq \begin{cases} \max\{p \in B_p^0; p \le wb_i\}, & \text{if } \sum_{1\le j\le M} qb_j \le C, \\ \min\{p \in B_p^0; p > Pb_i(vb_i, b_{-i})\}, & \\ \quad \text{if } wb_i \ne p_{\max} \text{ and } \sum_{1\le j\le M} qb_j > C, & \\ p_{\max}, \quad \text{if } wb_i = p_{\max} \text{ and } \sum_{1\le j\le M} qb_j > C, & \end{cases} \tag{21}$$

$p_{\max} \triangleq \max B_p^0$, and (vb_i, wb_i) corresponds to the best strategy without quantization constraints. □

The key features of sellers' quantized strategies in the double auctions are as follows (and buyers' strategies are considered similarly):

(1) Eq. (18) specifies the upper-bound of sellers' bid prices, which should be less than $p_u = \min_j\{pb_j \ge ps^*\}$, i.e., the minimum bid price of matched buyers (otherwise, zero utility results for an unmatched seller with a price greater than p_u).

(2) As presented in (19), when the aggregated supply is less than the potential quantity C, the sellers bid a defensive (higher) quantized price with respect to the best bid price ws_j, but if the aggregated supply is greater than C, the agents bid an aggressive (lower) quantize price compared with ws_j. This feature is inherited from both the ADQ-PSP algorithm and the UQ-PSP algorithm (Jia and Caines, 2010).

Lemma 3.1. *Subject to Hypotheses 2.1 and 2.3, the quantized strategies specified in (18) and (20) are truth-telling and optimal up to a quantized level.*

Sketch of proof: The quantized strategies specified in (18) and (20) approximate the corresponding best replies up to a quantized level. The bid in (18) is necessarily truthful. So, given s on the supply side, if γ is defined by

$$\begin{aligned} \gamma &= \max_{1\le j\le N}\{u_j((ws_j, v_j), s_{-j}) - u_j((ps_j, qs_j), s_{-j})\} \\ &\le \max_{j,(p,p')\subset B_p^0} |V_j(p) - V_j(p')|, \end{aligned} \tag{22}$$

where (p, p') is a pair of adjacent quantized prices, we see that the strategy (ps_j, qs_j) is γ-optimal. A similar argument applies on the demand side. □

This incentive-compatible property (i.e., a truth-telling strategy is dominant) is inherited from the VCG-like allocation rule (10)–(11) and (14)–(15). Lemma 3.1 implies that the strategies given in (18) and (20) are rational choices for agents in a quantized auction framework and therefore, we assume that

Hypothesis 3.2 *All agents in the double auctions specified in Section 2 apply the quantized strategies (18) and (20), respectively.*

Now we introduce the associated dynamical system for the considered double auction. Assume that all buyers and sellers do not have budget constraints and update their strategies *simultaneously* and *recursively*. Then given B_p^0, $\{\sigma_j\}_{1\le j\le N}$ and $\{\delta_i\}_{1\le i\le M}$, the recursive dynamical system for the double auction above consists of two subsystems:

Dynamical Quantized PSP double auction state-space system:

Buyer-side dynamical auction subsystem

$$\begin{aligned} vb_i^{k+1} &= \sup\{q \ge 0 : \delta_i(q) > Pb_i(q, b_{-i}^k)\}, \\ wb_i^{k+1} &= \delta_i(vb_i^{k+1}), \\ pb_i^{k+1} &= T_b(b^k, s^k), \\ qb_i^{k+1} &= D_i(pb_i^{k+1}), \end{aligned} \tag{23a}$$

Seller-side dynamical auction subsystem

$$\begin{aligned} vs_j^{k+1} &= \inf\left\{q \ge 0 : \sigma_i(q) > Ps_j(q, s_{-j}^k)\right\}, \\ ws_j^{k+1} &= \sigma_i(vs_j^{k+1}), \\ ps_j^{k+1} &= T_s(b^k, s^k), \\ qs_j^{k+1} &= I_j(ps_j^{k+1}), \end{aligned} \tag{23b}$$

Matched information update

$$pb^{*k+1} = \mathcal{B}^{k+1}(q^{*k+1}, b^{k+1}) \triangleq \mathcal{B}(q^{*k+1}, b^{k+1}), \tag{23c}$$

$$ps^{*k+1} = \mathcal{S}^{k+1}(q^{*k+1}, s^{k+1}) \triangleq \mathcal{S}(q^{*k+1}, s^{k+1}), \tag{23d}$$

$$C^{k+1} = C^{k+1}(b^{k+1}, s^{k+1}) \triangleq C(b^{k+1}, s^{k+1}), \tag{23e}$$

with $q^{*k+1} = \sup\{q : \mathcal{B}^{k+1}(q, b^{k+1}) \ge \mathcal{S}^{k+1}(q, s^{k+1})\}$, the initial conditions $pb_i^0 \in B_p^0$, $qb_i^0 = D_i(pb_i^0)$, $1 \le i \le M$, $ps_j^0 \in B_p^0$, $qs_j^0 = I_j(ps_j^0)$, $1 \le j \le N$, $C^0 = C(b^0, s^0)$, and $0 \le k < \infty$.

4. CONVERGENCE ANALYSIS

In this section, we study the convergence property of the dynamical double auction system (23). First, we define two quantized price sets

$$N_p^k \triangleq \left[pb_{\min}^k, ps_{\max}^k\right) \cap B_p^0, \quad k \ge 0, \tag{24}$$

$$M_p^k \triangleq \left[ps_{\min}^k, pb_{\max}^k\right] \cap B_p^0, \quad k \ge 0, \tag{25}$$

where

$$pb_{\min}^k = \min_{1\le i\le M}\{pb_i^k\}, \quad ps_{\min}^k = \min_{1\le j\le N}\{ps_j^k\},$$

$$pb_{\max}^k = \max_{1\le i\le M}\{pb_i^k\}, \quad ps_{\max}^k = \max_{1\le j\le N}\{ps_j^k\}.$$

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The notation can be found in the example in Figure 1. Since $pb^{*k} \geq pb^k_{\min}$ and $ps^{*k} \leq ps^k_{\max}$, $|N^k_p| = 0$ implies all agents are matched. M^k_p describes a minimum segment-wise quantized price set at Iteration k, which includes all agents' bid prices.

The convergence is established based upon the following three observations:

(i) For simplicity of analysis, we assume that the initial biding profile (b^0, s^0) is such that $C^0 > 0$. Then by the algorithm (23), it can be shown that, in order to achieve positive utilities (i.e., match guaranteed), buyers' bid prices should be greater than or equal to sellers' bid prices, i.e., $|N^k_p| = 0$ for all $k \geq 1$ (see Claims (a) and (b) in the proof of Lemma 4.4). That is to say, all agents are matched after the first iteration and hence $C^k > 0$ for all $k \geq 1$.

(ii) Before an agreement is achieved, the potential quantity C^k is chosen as the maximum of the matched quantities on both sides (based upon (7)) which consequentially increases the matched quantities in the next iteration under the condition $|N^k_p| = 0$ for all $k \geq 1$ in (i). The quantized price set M^k_p also shrinks (see Claims (c) and (d) in the proof of Lemma 4.4).

(iii) The minimum matched quantity is upper-bounded by the quantity corresponding to the intersection of the aggregate demand function and the aggregate supply function (see Theorem 4.3). We also show in Theorem 4.3 that the limit bid profiles of the dynamical double auction approximate the associated competitive equilibrium, which implies the optimization of social welfare under mild constraints on demand functions and supply functions (see Hypothesis 4.1).

Hypothesis 4.1. *There exist $\kappa_1 > 0$ and $\kappa_2 > 0$, such that all supply functions satisfy $\sigma_i(q) - \sigma_i(q') < \kappa_1(q - q')$, and all demand functions satisfy $\delta_i(q) - \delta_i(q') > -\kappa_2(q - q')$, for all $q > q' \geq 0$.*

Hypothesis 4.2. *(D-I price quantization hypothesis) We assume that the initial quantized price set, the aggregate demand function of all buyers, and the aggregate supply function of all sellers are such that for any two different quantized prices $p_1, p_2 \in B^0_p$,*

$$\sum_{1 \leq i \leq M} D_i(p_1) \neq \sum_{1 \leq i \leq N} I_i(p_2). \tag{26}$$

This *D-I* price quantization hypothesis excludes the occurrence of a deadlock due to quantization.

Theorem 4.3. *Subject to Hypotheses 3.2 and 4.2, for any initial bidding profile (b^0, s^0) such that $C^0 > 0$, the dynamical double auction system (23) converges at some $k^* < \infty$, to a non-trivial order-two orbit such that*

(i) *at Iteration k^*, $k^* \geq 0$, all sellers have the bid $(p^*_2, I_j(p^*_2))$, $1 \leq j \leq N$, and all buyers have the bid of $(p^*_1, D_i(p^*_1))$, $1 \leq i \leq M$;*

(ii) *at Iteration $(k^* + 1)$, all sellers and buyers have the bid pairs $(p^*, I_j(p^*))$, $1 \leq j \leq N$ and $(p^*, D_i(p^*))$, $1 \leq i \leq M$, respectively;*

(iii) *at Iteration $(k^* + 2)$, agents repeat the bid behaviours of the k^*th iteration, where p^*_1, p^*_2, p^* satisfy*

$$p^*_1 = \min\left\{ p \in B^0_p : \sum_{1 \leq i \leq M} D_i(p) \leq \sum_{1 \leq j \leq N} I_j(p) \right\}, \tag{27}$$

$$p^*_2 = \max\left\{ p \in B^0_p : \sum_{1 \leq i \leq M} D_i(p) \geq \sum_{1 \leq j \leq N} I_j(p) \right\}, \tag{28}$$

$$p^* = \arg \max_{p \in \{p^*_1, p^*_2\}} \left(\min\left\{ \sum_{1 \leq i \leq M} D_i(p), \sum_{1 \leq j \leq N} I_j(p) \right\} \right), \tag{29}$$

and are independent of the initial bidding profiles s^0 and b^0.

In particular, if $p^*_1 = p^*_2$ and $\sum_{1 \leq i \leq M} D_i(p^*_1) = \sum_{1 \leq i \leq N} I_i(p^*_2)$, then the system converges to a unique limit quantized price.

Furthermore, subject to Hypotheses 2.1, 2.3 and 4.1, the order-two orbit approximates the social optimum up to a quantized level.

Sketch of proof: Based upon the assumptions on demand functions and supply functions, there exists a unique price p_e satisfying

$$\sum_{1 \leq i \leq M} D_i(p_e) = \sum_{1 \leq i \leq N} I_i(p_e), \tag{30}$$

which is called the *competitive equilibrium price*. Subject to Hypotheses 2.1, 2.3 and 4.1, the social welfare function (i.e., the summation of (potential) valuation functions) is maximized at this equilibrium. One may check that p^*_1 is the smallest quantized price in the initial bid price set B^0_p greater than p_e, and p^*_2 is the largest quantized price in B^0_p less than p_e.

In Lemma 4.4 we show that, subject to Hypothesis 3.2 and 4.2, the dynamical double auction system (23) will converge to an order-two orbit, where two limit adjacent prices satisfy (27) and (28), and the potential quantity C^k converges to a unique value C^* then.

The special case where $p^*_1 = p^*_2 = p_e$, is analyzed in Appendix C. □

Figure 3 demonstrates the approximate competitive equilibrium nature of the limit order-two orbit for the quantized double auction.

Lemma 4.4. *Subject to Hypothesis 3.2 and 4.2, for any initial bidding profile such that $C^0 > 0$, the dynamical double auctions specified in Theorem 4.3 converges in time of order $O(|B^0_p|)$.*

Proof. See Appendix B. □

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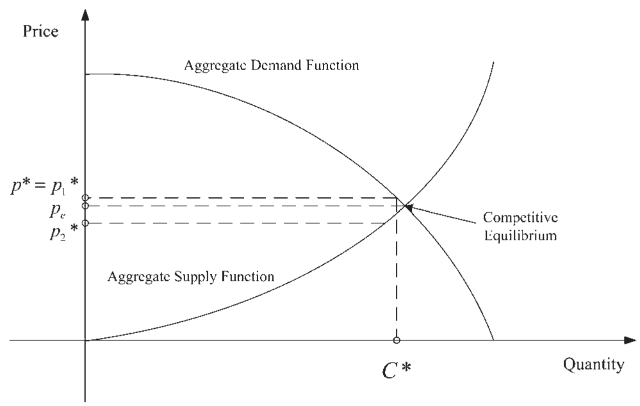

Figure 3. Quantized approximation of a competitive equilibrium

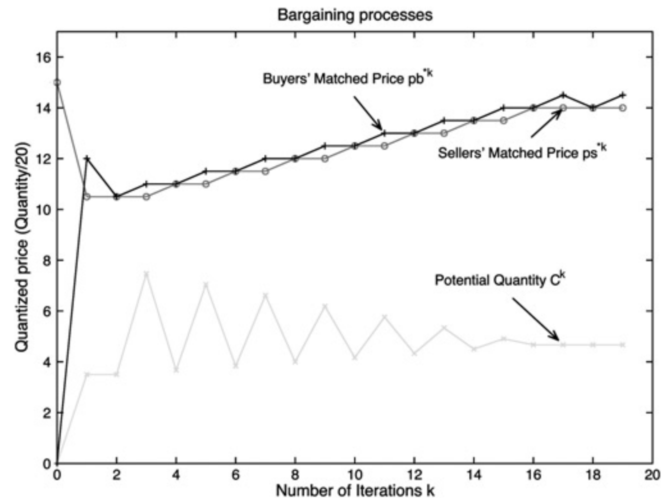

Figure 4. Convergence of a double-sided power auction with 20 buyers and 1 seller

In the order-two orbit, an auctioneer can be given a simple rule to choose a deal price $p^* \in \{p_1^*, p_2^*\}$ to break the bidding oscillation, where p^* given in (29) is independent of the initial bidding profiles s^0 and b^0. One may check that, the minimum matched quantity $C_{\min}^k = \min\{Cb^{*k}, Cs^{*k}\}$ achieves the maximum value C^* at p^*, which also corresponds to the maximum possible social utility subject to the quantized bid price set B_p^0. On the other hand, if $\sum_{1 \le i \le M} D_i(p_1^*) = \sum_{1 \le i \le N} I_i(p_2^*)$, the maximum social utility is achieved by either p_1^* or p_2^*.

5. NUMERICAL SIMULATIONS OF POWER AUCTIONS

A simple toy model for a power system is presented here to illustrate the results in Section 4. Five numerical examples are shown for the competition occurring on both sides of a power market, where all agents are assumed to apply the quantized strategies specified in Hypothesis 3.2. We assume that N power generators in a power market compete to provide electricity resource for competition among M users. For simplicity of calculation, supply functions σ_i, $1 \le i \le N$, are modelled as

$$\sigma_i(q) = \max((\alpha_4 + \alpha_3 \cdot q), 0). \tag{31}$$

Note here that the equation (31) is the derivative function of the "cost function", or the "marginal cost" discussed in work (Green and Newbery, 1992; Motto and Galiana, 2002; Peng *et al*., 2003). And demand functions δ_i, $1 \le i \le M$, are similarly given as

$$\delta_i(q) = \max((\alpha_2 - \alpha_1 \cdot q), 0).$$

Here $\alpha_1 - \alpha_4$ are design parameters.

In the first example, we assume that $N = 1$ and $M = 20$, that is to say, a single seller is considered to provide electricity resource for 20 users. Different from the single-sided auctions analyzed in Jia and Caines, 2010 where C is fixed, here the total quantity C changes according to the buyers' bidding profile. The dynamical behaviours of the matched prices and potential quantity are shown in Figure 4, which applies the data in Tables 1 and 2, and $B_p^0 = \{0.5 \cdot i\}_{1 \le i \le 30}$. It takes 16 steps for the auction to converge in this case, i.e., $k^* = 16$. Moreover, we can calculate $p_e = 14.2$, and then we may verify that $p_1^* = 14.5 > p_e > p_2^* = p^* = 14.0$.

TABLE 1.
Demand function parameters in Examples 1 and 3

Buyer #	1	2	3	4	5	6	7
α_1	1.24	0.91	0.92	1.03	0.80	1.45	1.45
α_2	23.22	15.25	19.14	22.31	22.81	18.67	22.45
Buyer #	8	9	10	11	12	13	14
α_1	1.24	1.04	1.53	1.25	1.66	1.59	1.06
α_2	23.92	17.43	16.30	17.25	18.50	17.87	24.27
Buyer #	15	16	17	18	19	$M = 20$	
α_1	1.25	1.05	1.32	1.50	1.45	0.83	
α_2	15.51	20.93	16.63	23.38	16.68	20.02	

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TABLE 2.
Single supply function parameters in Example 1

Seller #	$N = 1$
α_3	0.15
α_4	0

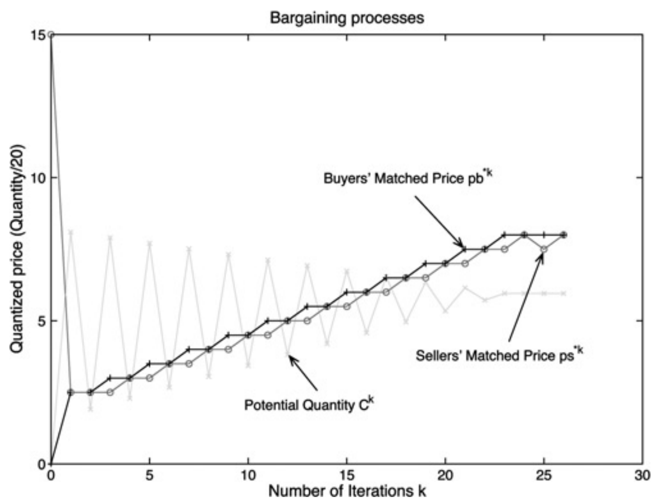

Figure 5. Convergence of a double-sided power auction with 1 buyer and 15 sellers

TABLE 3.
Single demand function parameters in Example 2

Buyer #	$M = 1$
α_1	0.13
α_2	23.30

In the second example, we assume that $N = 15$ and $M = 1$. The convergence property is shown in Figure 5, which applies the data in Tables 3 and 4, and $B_p^0 = \{0.5 \cdot i\}_{1 \leq i \leq 30}$. In this case, $k^* = 23$, $p_e = 7.88$, and it satisfies that $p_1^* = p^* = 8 > p_e > p_2^* = 7.5$, which matches the results in Theorem 4.3.

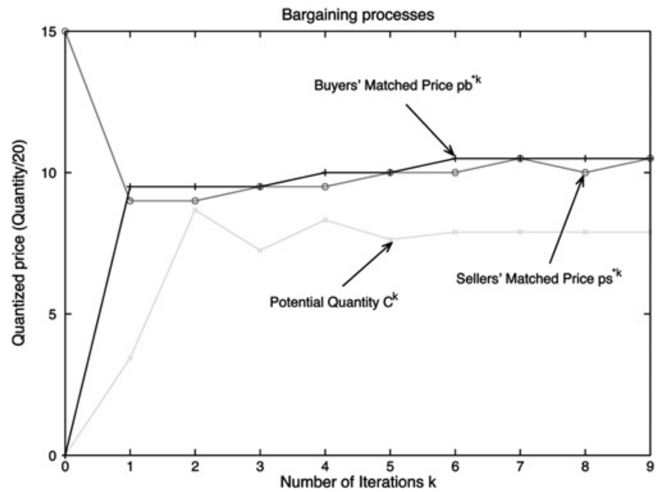

Figure 6. Convergence of a double-sided power auction with 20 buyers and 15 sellers

Example 3 considers a double auction for 15 sellers and 20 buyers, with the same data in Tables 1 and 4, and the same B_p^0. Figure 6 shows the dynamics of this example, where we may check $k^* = 6$, $p_1^* = 10.5 = p^*$, and $p_2^* = 10$. We also calculate $p_e = 10.43$, that is to say, the limit matched quantized prices p_1^* and p_2^* approximate p_e subject to the quantization assumption. Moreover, we can see that the convergence time for Example 3 is less than those for Examples 1 and 2. It may be conjectured that the competition among the sellers or the buyers accelerates the dynamical auction convergence as shown in Figures 4–6.

Figures 7 and 8 are presented for two more examples. We can observe both the rapid convergence and the order-two oscillations, which correspond to the best quantized approximations (from top and from bottom) for p_e, i.e., the prices associated with the competitive equilibria of the power markets. Also the convergence time for each example is less than the cardinality of the initial quantize price set, 30.

For all these simulations, we can observe the monotonic behaviours on matched prices pb^{*k} and ps^{*k} for some $k \geq 0$, which correspond to Claim (d) in the proof of Lemma 4.4: When $|M_p^k| \leq 2$ for some $k \geq 0$, all buyers bid $p_1 = pb^{*k}$ and all

TABLE 4.
Supply function parameters in Examples 2 and 3

Seller #	1	2	3	4	5	6	7	8
α_3	0.79	0.90	1.36	1.11	1.49	0.70	1.33	1.18
α_4	0	0	0	0	0	0	0	0
Seller #	9	10	11	12	13	14	$N = 15$	
α_3	0.75	0.98	0.90	1.10	1.30	0.60	1.32	
α_4	0	0	0	0	0	0	0	

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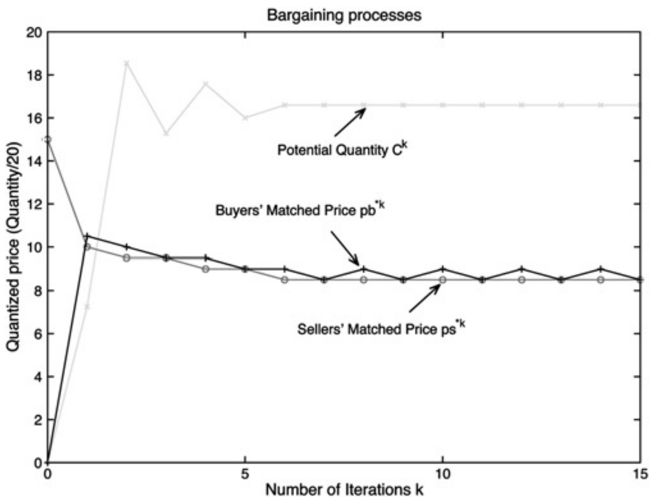

Figure 7. Convergence of a double-sided power auction with 30 buyers and 35 sellers

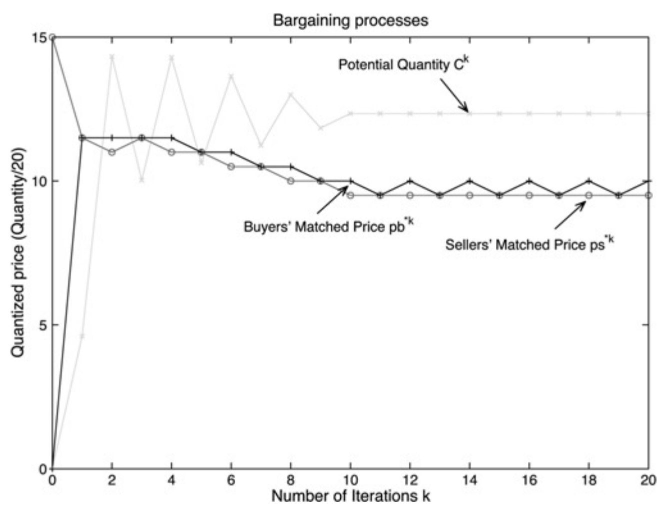

Figure 8. Convergence of a double-sided power auction with 30 buyers and 25 sellers

sellers bid $p_2 = ps^{*\mathrm{k}}$, where p_1 and p_2, $p_1 \geq p_2$, are two adjacent quantized prices or the same. p_1 and p_2 will move toward p_e together. The convergence of C^k is also observed in all examples.

6. CONCLUSIONS

In this paper, a novel auction algorithm has been applied to competitive electric power markets, motivated by the fact that restructuring an efficient market-based pricing electricity system has been occurring around the world. Considering the situation where competition may appear on both sides of the power markets, we have formulated the pricing problem as a quantized progressive second price double auction. Quantized strategies for such a double auction have been developed based upon our previous work of the ADQ-PSP algorithm and the UQ-PSP algorithm in single-sided auctions. We have proved that the dynamical double auction system associated with these iteratively updated quantized strategies converges rapidly to an order-two orbit, independent of initial data. Moreover, subject to mild assumptions on demand functions and supply functions, the limit orbit approximates the social optimum up to a quantized level.

So far, our work mainly focuses on the single-period competitive electricity market model. It is of interest and of potential value to apply the schemes developed in this paper to power pool auctions (see e.g., Madrigal and Quintana, 2001; Motto *et al*., 2002); in this case strategic behaviours of agents in a competitive multiple-period electricity market would be under consideration.

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APPENDIX A. LEMMA A

Consider the dynamical double auction system (23). Subject to Hypothesis 3.2, if

$$\min_{1\leq i\leq M}\{pb_i^k\} \geq \max_{1\leq j\leq N}\{ps_j^k\}, \tag{32}$$

for some $k \geq 0$, then

$$\min_{1\leq i\leq M}\{pb_i^{k+t}\} \geq \max_{1\leq j\leq N}\{ps_j^{k+t}\}, \tag{33}$$

for all $t \geq 0$.

Proof. The result is directly from Equation (18) and its dual equation on the demand side. Precisely, if (32) holds, we have $pb^{*\mathrm{k}} \geq ps^k_{\max}$ and $ps^{*\mathrm{k}} \leq pb^k_{\min}$. Then for any buyer,

$$pb_i^{k+1} \geq \max_j\{ps_j^k \leq pb^{*\mathrm{k}}\} = ps^k_{\max},$$

and for any seller

$$ps_j^{k+1} \leq \min_i\{pb_i^k \geq ps^{*\mathrm{k}}\} = pb^k_{\min},$$

from (18).

Furthermore, when (32) holds,

$$C^k = \max\left\{\sum_{1\leq i\leq M} qb_i^k, \sum_{1\leq j\leq N} qs_j^k\right\}.$$

If $C^k = \sum_{1\leq i\leq M} qb_i^k$, $pb_i^{k+1} \geq pb^k_{\min}$ for any buyer i, $1 \leq i \leq M$, and then (33) holds. If $C^k = \sum_{1\leq j\leq N} qs_j^k$, we can check in a similar way that (33) holds. □

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APPENDIX B. SKETCH PROOF OF LEMMA 4.4

By defining two quantized price sets N_p^k and M_p^k with (24) and (25) respectively, we claim that

(a) For any initial state such that $C^0 > 0$, $|N_p^1| = 0$, that is to say, all agents in the auction are matched (i.e. have positive potential allocated quantities) after the first iteration.
(b) When $|N_p^k| = 0$, $|N_p^{k+1}| = 0$, that is to say, the agents' recursive strategies guarantee their matched status remains unchanged in the dynamical double auction.
(c) When $|N_p^k| = 0$,

$$|M_p^{k+1}| < |M_p^k|, \quad \forall |M_p^k| > 2, \tag{34}$$

i.e., the set of agents' bid prices monotonically shrinks until it consists of at most two prices, if all agents are matched.
(d) When $|M_p^k| = 2$ or $|M_p^k| = 1$, the dynamical double auction converges in at most $2|B_p^0|$ iterations.

Proof of Claim (a)

First we assume that $C^0 = Cb^*(b^0, s^0)$. Subject to the quantized strategies specified in Hypothesis 3.2, each buyer B_i, $1 \le i \le M$, will bid a price $pb_i^1 \ge \min\{pb_i^0; pb_i^0 \ge ps^{*0}\}$ (since all intersections between demand functions and buyer market price functions should be on or above $p = \min\{pb_i^0; pb_i^0 \ge ps^{*0}\}$), and each seller S_j, $1 \le j \le N$, will bid a price $pb_j^1 \le \min\{pb_i^0; pb_i^0 \ge ps^{*0}\}$ from (18). Then we have $pb_{\min}^1 \ge ps_{\max}^1$, which implies that $|N_p^1| = 0$.

If $C^0 = Cs^*(b^0, s^0)$, $|N_p^1| = 0$ is also achieved in the similar way.

Proof of Claim (b)

See Lemma A.

Proof of Claim (c)

When $|N_p^k| = 0$, it implies that $pb_{\min}^k \ge ps_{\max}^k$. Together with the definition of C^k, we have $C^k = \max\{\sum_{1\le i\le M} qb_i^k, \sum_{1\le j\le N} qs_j^k\}$. By checking the quantized strategies given in Hypothesis 3.2, we see that (I) $pb_{\max}^k \ge pb_{\max}^{k+1}$, and furthermore (II) $pb_{\max}^k > pb_{\max}^{k+1}$ if either (II.1) $pb_{\max}^k > pb_{\min}^k$; or (II.2) $pb_{\max}^k = pb_{\min}^k$ and $\sum_{1\le i\le M} qb_i^k < \sum_{1\le j\le N} qs_j^k$. The similar argument holds for the seller-side. Therefore, subject to Hypothesis 4.2 and $|M_p^k| > 2$, in each iteration either the upper bound of M_p^k decreases, or the lower bound of M_p^k increases, or both happen. That implies (34).

Proof of Claim (d)

When $|M_p^k| = 2$ or $|M_p^k| = 1$, it is the case that all buyers bid the same price p_1 and all sellers bid the same price p_2, where $p_1 \ge p_2$ are two adjacent quantized bid prices. (If $p_1 = p_2$, $|M_p^k| = 1$, otherwise, $|M_p^k| = 2$). By checking the corresponding dynamics, p_1 and p_2 will move toward the competitive equilibrium price p_e together in the following iterations.

Precisely, if $p_2 > p_e$, we may verify (from the definitions of p_e in (30), demand functions and supply functions in Section 2) that $\sum_{1\le j\le N} I_j^k(p_2) > \sum_{1\le i\le M} D_i^k(p_1)$. On the buyer-side, $pb_i^{k+1} = p_2$ for all $1 \le i \le M$; and on the seller side, if $p_1 > p_2$, $ps_j^{k+1} = p_2$ for all $1 \le j \le N$, or if $p_1 = p_2$, $ps_j^{k+1} = \max\{p_i \in M_p^k : p_i < p_2\}$ for all $1 \le j \le N$. Similarly, if $p_1 < p_e$, all buyers or all sellers will bid a higher price iteratively until $p_1 \ge p_e$ and $p_2 \le p_e$, at which stage the order-two orbit oscillation will begin (we can check that $p_1 = p_1^*$ and $p_2 = p_2^*$ as defined in (27) and (28)).

The last step is to check that once all agents are in the order-two orbit, they will stay there for all the following iterations.

Define $C_{\min}^k = \min\{Cb^k, Cs^k\} = \min\{\sum_{1\le i\le M} qb_i^k, \sum_{1\le j\le N} qs_j^k\}$. When $pb_i^k = p_1^*$ and $ps_j^k = p_2^*$, if $C_{\min}^k = Cb^k$, subject to Hypothesis 4.2, we have $C^k = Cs^k > Cb^k$, which implies that all buyers will bid $pb_i^{k+1} = p_2^*$ and all sellers will stay at p_2^* at the $(k+1)$th iteration. We can verify that $p_2^* = p^*$ as defined in (29). At the $(k+1)$th iteration, we can check $C_{\min}^{k+1} = Cs^{k+1} = \sum_{1\le j\le N} qs_j^{k+1}$ and $C^{k+1} = C_{\min}^{k+1} = C^k$ from $B_p^{k+1} = \{p_2^*\}$ and (7). As a result, all buyers will bid aggressively and choose p_1^* as their bid price at the $(k+2)$th iteration, but all sellers will still stay at p_2^*. Therefore, the dynamical auction system will oscillate between the two states (i.e., the state at the kth iteration and the state at the $(k+1)$th iteration) and C^k converges. If $C_{\min}^k = Cs^k$, the same result can be achieved.

Overall, since in every two iterations the upper bound (or the lower bound) of M_p^k decreases (or increases) one price level until the dynamics converge to the order-two orbit, the convergence time is bounded by $2|B_p^0|$.

Furthermore, in the worst case, $|M_p^k| = |B_p^0|$. Based upon Claims (a)–(d), one may check that the time for the dynamical double auction system (23) to converge to the order-two orbit is of order $O(|B_p^0|)$. □

APPENDIX C. CONVERGENCE ANALYSIS FOR THE SPECIAL CASE $P_1^* = P_2^* = P_e$

For the special case that $p_1^* = p_2^* = p_e$, i.e., $\sum_{1\le i\le M} D_i(p_1^*) = \sum_{1\le j\le N} I_j(p_2^*)$, we can check that all buyers and sellers will stay in $p^* = p_e$ following Claim (d) in the proof of Lemma 4.4: since $p_1 \ge p_e$ and $p_2 \le p_e$ for a converged bid set $M_p^k = \{p_1, p_2\}$, if $p_1 = p_2 = p_e$, $C^k = C_{\min}^k = \sum_{1\le i\le M} qb_i^k = \sum_{1\le j\le N} qs_j^k$, and by checking the quantized strategies defined in Hypothesis 3.2, we have all buyers and sellers will bid p_e at the next iteration. Hence the system (23) converges to p_e in this case.

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